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A STRUCTURE THEOREM FOR HOLOMORPHIC CURVES IN $\mathrm{Gr}(3, \mathbf{C}^6)$

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Abstract

A holomorphic curve f in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is called generic if the curvature of the canonical connection of $f^*(S(n, \mathbf{C}^{2n}))$ has distinct eigenvalues, where $S(n, \mathbf{C}^{2n})$ is the universal subbundle over $\mathrm{Gr}(n, \mathbf{C}^{2n})$. A holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is completely split if it is the orthogonal direct sum of n holomorphic curves in the projective plane. These two types of curves are both relatively simple. In this paper, we prove that a 1-nondegenerated holomorphic curve in $\mathrm{Gr}(3, \mathbf{C}^6)$ is either generic or completely split.

Introduction

Denote the Grassmannian of n-dimensional subspaces of $\mathbf{C}^{2n}$ by $\mathrm{Gr}(n, \mathbf{C}^{2n})$. A holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is locally a holomorphic mapping of some open disk in $\mathbf{C}$ into $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Because of the analytic structure, we can restrict ourselves to the local holomorphic curves only.

Let $f: \Omega \to \mathrm{Gr}(n, \mathbf{C}^{2n})$ be a holomorphic curve. For each z in Ω, we define $(f(z), f'(z)) = \mathrm{span}\{\gamma_1(z), \cdots, \gamma_n(z), \gamma_1'(z), \cdots, \gamma_n'(z)\}$, where $\gamma_j: \Omega \to \mathbf{C}^{2n}$ is holomorphic and $\mathrm{span}\{\gamma_1(z), \cdots, \gamma_n(z)\} = f(z)$. Clearly, (f, f') is independent of the choice of $\gamma_1, \cdots, \gamma_n$. We say f is 1-nondegenerated if $(f(z), f'(z)) = \mathbf{C}^{2n}$ for each $z \in \Omega$.

Throughout this paper, by "holomorphic curve" we mean "1-nondegenerated holomorphic curve". Let f be a 1-nondegenerated holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Then the holomorphic Hermitian vector bundle

$$E_f: \begin{array}{c}\text{the space } f(z)\\ \downarrow\\ z\end{array}$$

is a completely unitary invariant of f by the Calabi rigidity theorem. By

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the unitary equivalence of f_1 and f_2 we mean that there is a unitary transformation U of $\mathbf{C}^{2n}$ making $U \cdot f_1 = f_2$. We shall name the canonical connection of E_f and its curvature the connection of f and the curvature of f, respectively.

Definition 1. A holomorphic curve is called *generic* if its curvature has distinct eigenvalues at some point.

In [3] and [1], it was proved that a second order contact of two generic curves implies unitary equivalence. In this paper we shall prove that any holomorphic curve in $\mathrm{Gr}(3, \mathbf{C}^6)$ is either generic or an orthogonal direct sum of three holomorphic curves in the projective plane. Thus in $\mathrm{Gr}(3, \mathbf{C}^6)$, two holomorphic curves having second order contact must be unitarily equivalent, which answers the so-called Griffiths' conjecture in the simplest nontrivial case.

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Main results

Let f be a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Using the canonical coordinate of the Grassmannian, we see locally E_f has the columns of $\binom{I}{P}$ as a holomorphic frame, where I is the $n \times n$ identity matrix and P is an $n \times n$ matrix of analytic function entries. Over this holomorphic frame, the matrix of the curvature bundle map K_f ($K_f dz d\overline{z}$ is the curvature tensor) is

$$-(I + P^* P)^{-1} P'^* (I + PP^*)^{-1} P'.$$

A quick consequence of this expression is that the eigenvalues of K_f are all strictly negative.

From the above expression, it follows that

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \det(-K_f) = -2 \frac{\partial^2}{\partial z \partial \overline{z}} \log \det(I + P^* P).$$

By a well-known lemma of S. S. Chern, $\operatorname{tr} K_f = K_{\bigwedge^n(E_f)}$. Noting that

$$K_{\bigwedge^n(E_f)} = -\frac{\partial^2}{\partial z \partial \overline{z}} \log \det(I + P^* P) = \frac{1}{2} \frac{\partial^2}{\partial z \partial \overline{z}} \log \det(-K_f),$$

we thus have shown

Lemma 1. *Let* $\lambda_1, \cdots, \lambda_n$ *be the smooth eigenfunctions of* K_f *and define*

$$f(\lambda_i) = 2\lambda_i^2 - \lambda_i \frac{\partial^2}{\partial z \partial \overline{z}} \log(-\lambda_i).$$

Then

$$\sum_{i=1}^{n} F(\lambda_i)/\lambda_i = 0.$$

Definition 2. We say a holomorphic curve f is completely split if E_f is an orthogonal direct sum of n holomorphic line bundles. Equivalently, f is an orthogonal direct sum of n holomorphic curves in the projective plane.

Our first aim is to show:

"f is completely split $\Leftrightarrow F(\lambda_i) = 0$ for all i".

In order to do this, we need to look back at the differential structure on E_f.

Recall that a bundle map of E to E is a C^∞ map which maps each fiber linearly to itself. Let φ be a bundle map of E_f to E_f, where f is a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Then we define

$$[D, \varphi] = D \circ \varphi - (\varphi \otimes \mathrm{id}) \circ D = \varphi_z\, dz + \varphi_{\overline{z}}\, d\overline{z}.$$

Although D is not a bundle map, a quick check gives that φ_z and $\varphi_{\overline{z}}$ are all bundle maps of E_f to E_f. We call them the first covariant derivatives of φ. So $\varphi_{z\overline{z}}$ would be one of the first covariant derivatives of $\varphi_{\overline{z}}$.

If over an orthonormal frame S the connection matrix is $\Theta dz - \Theta^* d\overline{z}$, then

$$\varphi_z(S) = [\Theta, \varphi(S)] + \frac{\partial \varphi(S)}{\partial z},$$
$$\varphi_{\overline{z}}(S) = [-\Theta^*, \varphi(S)] + \frac{\partial \varphi(S)}{\partial \overline{z}}.$$

For details, we refer the reader to [2]. Also in [2] it was proved that $K_{\overline{z}z} = K_{z\overline{z}}$ (write K_f as K), although $K_{z^2\overline{z}} \neq K_{\overline{z}z^2}$ in general.

In [1], the following was proved:

(*) "an n-dimensional Hermitian holomorphic vector bundle is equivalent to some E_f with f a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n}) \Leftrightarrow 2K^2 + K_{\overline{z}}K^{-1}K_z = K_{z\overline{z}}$."

Definition 3. An orthonormal frame is called a first adapted frame if, over it, the matrix of K_f is smoothly diagonalized.

From now on K will stand for the matrix of the curvature.

Now we are ready to show

Theorem 1. *A holomorphic curve f in* $\mathrm{Gr}(n, \mathbf{C}^{2n})$ *is completely split* $\Leftrightarrow F(\lambda_i) = 0$ *for all i, where* $\lambda_1, \cdots, \lambda_n$ *are the smooth eigenfunctions of* K_f.

Proof. The forward direction is trivial. For the backward direction, we need the following fact from [2] to reduce the problem: a holomorphic curve is completely split iff over some first adapted frame $[\Theta, K] \equiv 0$ where $(\Theta dz - \Theta^* d\overline{z}$ is the connection matrix. Then, take any first adapted frame and write K_z, $K_{\overline{z}}$, $K_{z\overline{z}}$ in matrix form:

$$K_z = [\Theta, K] + \frac{\partial K}{\partial \overline{z}}, \qquad K_{\overline{z}} =]\Theta^*, K] + \frac{\partial K}{\partial \overline{z}},$$
$$K_{z\overline{z}} = [-\Theta^*, [\Theta, K]] + \left[-\Theta^*, \frac{\partial K}{\partial z}\right] + \frac{\partial}{\partial \overline{z}}[\Theta, K] + \frac{\partial^2}{\partial z \partial \overline{z}} K.$$

Substituting them into $(*)$ and taking the trace on both sides, we have

$$\sum_{i=1}^{n} F(\lambda_i) + \mathrm{tr}[\Theta, K]^* K^{-1}[\Theta, K] \equiv 0,$$

i.e., $\mathrm{tr}[\Theta, K]^* K^{-1}[\Theta, K] \equiv 0$. Since K^{-1} is negative definite, we obtain $[\Theta, K] \equiv 0$.

Now, we can direct our attention to our final aim. We assume there is a nongeneric curve f, which is not completely split, and fix it once and for all. We shall then use the following three steps to obtain a contradiction.

Let us assume that over some first adapted frame the curvature matrix is

$$K = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix}$$

with $\lambda \neq \mu$. By Theorem 1 and Lemma 1 above, we may assume $F(\lambda) \neq 0$ and $F(\mu) \neq 0$. Let

$$\Theta = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

and assume

$$[\Theta, K] = \begin{pmatrix} 0 & a_{12}(\mu - \lambda) & a_{13}(\mu - \lambda) \\ a_{21}(\lambda - \mu) & 0 & 0 \\ a_{31}(\lambda - \mu) & 0 & 0 \end{pmatrix} \neq 0.$$

Without loss of generality, we take $|a_{12}|^2 + |a_{13}|^2 \neq 0$.

Step 1. There is a first adapted frame such that $a_{12} > 0$, $a_{31} > 0$ and $a_{21} \equiv a_{13} \equiv 0$.

Observe that if S is a first adapted frame and

$$K(S) = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix},$$

then for any $\left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)$, where U is a 2×2 C^∞ unitary matrix, $S\left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)$ is again a first adapted frame. But Θ changes to

$$\begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \Theta \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & U^* \partial U/\partial z \end{pmatrix},$$

so we may choose U smoothly, such that $a_{12} > 0$, $a_{13} \equiv 0$ and $a_{31} \geq 0$. Now

$$[\Theta, K] = \begin{pmatrix} 0 & a_{12}(\mu - \lambda) & 0 \\ a_{21}(\lambda - \mu) & 0 & 0 \\ a_{31}(\lambda - \mu) & 0 & 0 \end{pmatrix}.$$

Equation $(*)$ can be rewritten as

$$(**) \quad \begin{aligned} &\begin{pmatrix} F(\lambda) & & \\ & F(\mu) & \\ & & F(\mu) \end{pmatrix} + [\Theta, K]^* K^{-1} [\Theta, K] \\ &+ [\Theta, K]^* \begin{pmatrix} \frac{\partial \log \lambda}{\partial z} & & \\ & \frac{\partial \log \mu}{\partial z} & \\ & & \frac{\partial \log \mu}{\partial \mu} \end{pmatrix} \\ &+ \begin{pmatrix} \frac{\partial \log \lambda}{\partial \bar{z}} & & \\ & \frac{\partial \log \mu}{\partial \bar{z}} & \\ & & \frac{\partial \log \mu}{\partial \bar{z}} \end{pmatrix} [\Theta, K] \\ &= [-\Theta^*, [\Theta, K]] + \left[-\Theta^*, \frac{\partial K}{\partial z} \right] + \frac{\partial}{\partial \bar{z}} [\Theta, K], \end{aligned}$$

where

$$[\Theta, K]^* K^{-1} [\Theta, K] = \begin{bmatrix} \frac{(\lambda-\mu)^2}{\mu}(|a_{21}|^2 + |a_{31}|^2) & 0 & 0 \\ 0 & a_{12}^2 \frac{(\lambda-\mu)^2}{\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[-\Theta^*, [\Theta, K]] = \begin{bmatrix} (a_{12}^2 + |a_{21}|^2 + a_{31}^2)(\mu - \lambda) & a_{12}(\mu - \lambda)(\bar{a}_{22} - \bar{a}_{11}) & a_{12}\bar{a}_{32}(\mu - \lambda) \\ a_{21}(\bar{a}_{22} - \bar{a}_{11})(\mu - \lambda) + \bar{a}_{32} a_{31}(\mu - \lambda) & (|a_{21}|^2 + a_{12}^2)(\lambda - \mu) & a_{21}\bar{a}_{31}(\lambda - \mu) \\ a_{31}(\bar{a}_{33} - \bar{a}_{11})(\mu - \lambda) + a_{21}\bar{a}_{23}(\mu - \lambda) & a_{31}\bar{a}_{21}(\lambda - \mu) & a_{31}^2(\lambda - \mu) \end{bmatrix}.$$

Considering the $(2, 3)$ entry of $(**)$, we have

$$\frac{F(\mu)}{\lambda-\mu} = a_{31}^2.$$

Therefore $a_{31} > 0$ and $a_{21} \equiv 0$.

Step 2. $\lambda/\mu =$constant.

Fix the first adapted frame from Step 1 and consider the $(1, 3)$ and $(2, 1)$ entries of $(**)$. Then we get

$$\overline{a}_{32} = \frac{a_{31}}{a_{12}} \frac{\partial \log((\lambda-\mu)/\mu)}{\partial z}, \qquad \overline{a}_{32} = \frac{a_{12}}{a_{31}} \frac{\partial \log(\lambda/(\lambda-\mu))}{\partial z},$$

which are combined to give

$$(\dagger) \qquad \frac{a_{31}^2}{a_{12}^2} \frac{\partial \log((\mu-\lambda)/\mu)}{\partial z} = \frac{\partial \log(\lambda/(\mu-\lambda))}{\partial z}.$$

Now from the $(2, 2)$ entry of $(**)$ it follows that

$$a_{12}^2 = \frac{F(\mu)}{\lambda-\mu} \cdot \frac{\lambda}{\mu},$$

so that $a_{31}^2/a_{12}^2 = \mu/\lambda$. Substituting it into $(\dagger)$, we get

$$\frac{\partial \log(\lambda/(\mu-\lambda))}{\partial z} = \frac{\mu}{\lambda} \frac{\partial \log((\mu-\lambda)/\mu)}{\partial z},$$

i.e., $\partial \frac{\lambda}{\mu}/\partial z = 0$, or $\lambda/\mu =$ constant.

Let $\lambda/\mu = c$; then $c \neq 1$, $c > 0$. Thus $F(\lambda)/\lambda + 2F(\mu)/\mu = 0$ becomes

$$2\left(\frac{c+2}{3}\right)\mu = \frac{\partial^2 \log \mu}{\partial z \partial \overline{z}}.$$

Moreover, $a_{32} = 0$, $a_{31}^2 = -\frac{2}{3}\mu$, $a_{12}^2 = -\frac{2}{3}c\mu$.

Step 3. $a_{23} = 0$ and $a_{23} \neq 0$ (a contradiction).

Considering the $(3, 1)$ and $(1, 2)$ entries of $(**)$, we get

$$a_{33} - a_{11} = \frac{1}{2}\frac{\partial}{\partial z} \log \mu, \qquad a_{11} - a_{22} = \frac{1}{2}\frac{\partial}{\partial z} \log \mu,$$

addition of which gives $a_{33} - a_{22} = (\partial/\partial z) \log \mu$. Recall that K and Θ have to satisfy the connection-curvature equation:

$$(\Delta) \qquad -K = [\Theta, \Theta^*] + \frac{\partial \Theta}{\partial \overline{z}} + \frac{\partial \Theta^*}{\partial z},$$

where

$$[\Theta, \Theta^*] = \left[\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}, \begin{pmatrix} \overline{a}_{11} & 0 & a_{31} \\ a_{12} & \overline{a}_{22} & 0 \\ 0 & \overline{a}_{23} & \overline{a}_{33} \end{pmatrix}\right]$$

$$= \begin{pmatrix} a_{12}^2 - a_{31}^2 & a_{12}(\overline{a}_{22} - \overline{a}_{11}) & a_{31}(a_{11} - a_{33}) \\ a_{12}(a_{22} - a_{11}) & |a_{23}|^2 - a_{12}^2 & a_{23}(\overline{a}_{33} - \overline{a}_{22}) \\ a_{31}(\overline{a}_{11} - \overline{a}_{33}) & \overline{a}_{23}(a_{33} - a_{22}) & a_{31}^2 - |a_{23}^2| \end{pmatrix}.$$

Considering the $(2, 2)$ and $(3, 3)$ entries of (Δ), we have

$$-\mu = |a_{23}|^2 - a_{12}^2 + 2\,\mathrm{Re}\,\frac{\partial a_{22}}{\partial \overline{z}}, \qquad -\mu = a_{31}^2 - |a_{23}^2| + 2\,\mathrm{Re}\,\frac{\partial a_{33}}{\partial \overline{z}},$$

subtraction of which yields

$$2|a_{23}|^2 = a_{12}^2 + a_{31}^2 + 2\,\mathrm{Re}\,\frac{\partial (a_{33} - a_{22})}{\partial \overline{z}}$$

$$= \left(-\frac{2c}{3} - \frac{2}{3}\right)\mu + 2\frac{\partial^2 \log \mu}{\partial z \partial \overline{z}} = \frac{2}{3}(c+3)\mu.$$

Thus $a_{23} \neq 0$, since $c > 0$.

Next, consider the $(2, 3)$ entry of (Δ):

$$0 = a_{23}(\overline{a}_{33} - \overline{a}_{22}) + \partial a_{23}/\partial \overline{z}.$$

Since $a_{23} \neq 0$, if we write $a_{23} = |a_{23}|e^{i\theta}$, then

$$\frac{\partial (\log \mu + i\theta)}{\partial \overline{z}} = -\frac{\partial \log |a_{23}|}{\partial \overline{z}} = -\frac{1}{2}\frac{\partial \log \mu}{\partial \overline{z}},$$

or $(\partial/\partial \overline{z})[\log(-\mu)^{3/2} + i\theta] \equiv 0$, which implies

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log(-\mu) = 2\left(\frac{c+2}{3}\right)\mu \equiv 0,$$

a contradiction.

Because of this contradiction, we have reached

Theorem 2. *A holomorphic curve in* $\mathrm{Gr}(3, \mathbf{C}^6)$ *is either generic or completely split.*

References

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